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The difference between monopole vortices in planetary flows and laboratory experiments

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This work is an attempt to explain observations of vortices in experiments with shallow water in rotating paraboloidal vessels. The most long-lived vortices are invariably anticyclones, while cyclones quickly disperse, and they are larger than the Rossby radius. These experiments are designed to simulate geophysical flows, where large, long-lived, anticyclonic vortices are common.

The general condition for vortices to be steady is that they propagate faster than linear Rossby waves, so that the vortex energy is not dispersed by coupling to linear waves. The propagation velocity is determined by a general integral relation that gives the velocity of the centre of mass. In geophysical flows, to lowest order in the Rossby number, the difference between the centre-of-mass velocity and the maximum phase velocity of the Rossby waves is proportional to the relative perturbation of the fluid depth. Since for anticyclones the difference is positive they may be steady, whereas cyclones cannot be.

In the laboratory experiments this velocity difference is absent because of the latitudinal dependence of the effective gravity caused by the centrifugal force. However, to the next order in the Rossby number, there is another nonlinear contribution, so that anticyclones (but not cyclones) still propagate faster than the linear Rossby waves, and may thus be steady. The velocity difference is smaller than for geophysical flows, and vanishes in the limit of small Rossby number. The existence conditions also show that we can expect the experimental vortices to be smaller (as measured by the Rossby radius) than the planetary vortices. The theory does not apply to vortices that are much smaller than the Rossby radius.

1. Introduction

Large, long-lived vortices are a persistent feature of many geophysical flows. Perhaps the most well-known one is the Great Red Spot of Jupiter, which has existed for at least 300 years. There are also many other large vortices on Jupiter (for instance the Large Ovals, which have been known for several decades (Smith *et al.* 1979)) and on other planets, such as Saturn (Big Bertha, the Brown Spot, Anne's Spot, etc. (Smith *et al.* 1982)) and Neptune (the newly discovered Great Dark Spot (Smith *et al.* 1989)). In the oceans of the Earth, compact intrathermocline anticyclones can exist for several years, transporting trapped fluid over thousands of kilometres (McWilliams 1985).

A number of laboratory experiments have been performed to simulate such planetary flows. They are all done in rotating vessels, so that the dynamics is

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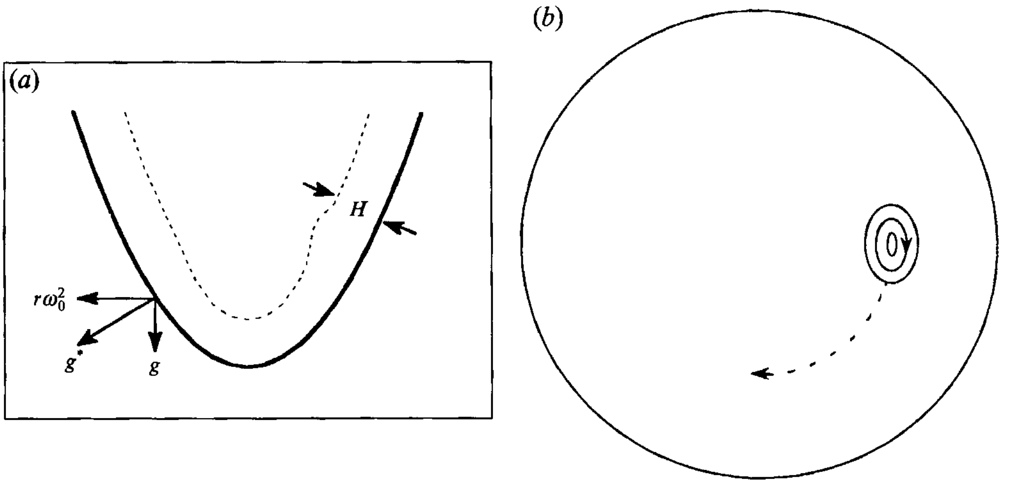

FIGURE 1. An experiment with paraboloidal vessel. (*a*) Vertical section. The dashed curve is the water surface. On the right-hand side the elevated surface of an anticyclonic vortex is seen. (*b*) Top view. An anticyclonic vortex (which is approximately circular when viewed perpendicularly to the water surface) is propagating westward.

dominated by the Coriolis force, but otherwise widely different designs are used. Some experiments are meant to simulate the generation of vortices by shear flows or thermal convection, while others demonstrate the possibility of long-lived free vortices. In the present paper we will focus on the latter.

An important empirical fact about the naturally occurring vortices is the cyclone–anticyclone asymmetry. The largest and most long-lived vortices are predominantly anticyclones. (This is particularly true on the large planets, but there are also exceptions, such as cyclonic Gulf Stream rings.) An explanation of this difference was recently proposed by Nycander & Sutyrin (1992). Basically, it could be traced to an exact integral relation of the shallow-water equations. When expanded to lowest order in the Rossby number, while keeping the relative perturbation $\Delta H/H$ of the fluid depth arbitrary, this relation gives the velocity of the centre of mass of any localized structure. It shows that anticyclones propagate westward faster than the linear Rossby waves, while the velocity of cyclones is within the region of linear phase velocities. (This will be referred to as the 'nonlinear β-effect', to emphasize the fact that the velocity difference vanishes if the equations are expanded to lowest order in $\Delta H/H$, as in the quasi-geostrophic approximation.) Thus, cyclones, but not anticyclones, loser energy by radiating linear waves.

The only experiments in which a cyclone–anticyclone asymmetry similar to that observed in nature could be clearly seen are those by Nezlin *et al.* (1990). They were performed in a shallow layer of water with a free surface in a rotating paraboloidal vessel, cf. figure 1. Because of the curvature of the bottom of the vessel, the effective Coriolis force decreases with the distance from the rotation axis, mimicking the β-effect on a planet. Single vortices were externally excited in the water, and then observed as they drifted along. It was found that anticyclones were more easily excited than cyclones, and that they survived for a longer time and travelled somewhat faster. The vortex radius was larger than or approximately equal to the Rossby radius, and the relative perturbation of the fluid depth was of order unity.

These results are seemingly in good agreement with the theory of Nycander & Sutyrin (1992) for planetary flows, at least qualitatively. Nevertheless, there is an important effect which is present in the laboratory experiments, but not in planetary flows, and which can be expected *a priori* to be of the same magnitude as the β-effect. This is the variation of the effective gravity g^* with the latitude (here called the 'γ-effect').

It will be shown here that this γ-effect exactly cancels the nonlinear β-effect to lowest order in the Rossby number ϵ. Thus, to this order in ϵ, the centre-of-mass velocity is independent of the amplitude, and coincides with the phase velocity of long-wavelength Rossby waves. The difference between cyclones and anticyclones has disappeared, and both would radiate linear waves.

This result seems to contradict the experimental evidence cited above. However, the analysis is based on an expansion in three small parameters: the shallow-water parameter δ (the ratio between the depth of the water and the radius of the vortex), the Rossby number ϵ (the ratio between the rotation frequency of the vortex and that of the vessel) and the size parameter α (the ratio between the radius of the vortex and the radius of curvature of the bottom surface). Actually, none of these parameters was very small in the experiments. Since the crucial nonlinear contribution to the centre-of-mass velocity exactly vanishes to lowest order in these parameters, it is reasonable to calculate higher-order contributions.

In the present paper, the calculation is therefore carried out to second order in the Rossby number, and an important new contribution to the centre-of mass velocity is found. This term shows that anticyclones indeed propagate faster than the linear Rossby waves, while the velocity of cyclones is within the region of linear phase velocities. Thus, the situation is, after all, qualitatively the same as for planetary flows, although the effect is less pronounced.

2. General conditions for steady vortices

In a wide variety of nonlinear wave equations, necessary conditions for the existence of localized, steady solutions can be found by a simple two-step analysis (Nycander & Pavlenko 1991). In the first step the linear dispersion relation is calculated. The reason for this is that the amplitude of any localized solution must be small far away from the centre, and the localization properties are therefore determined by the linearized equations. In general, any structure propagating with a velocity that coincides with the phase velocity of some linear wave will radiate energy, analogously to Cerenkov radiation, and gradually disperse. In order to be steady, localized nonlinear solutions should therefore propagate with a velocity outside the region of linear phase velocities. The amplitude then decreases exponentially away from the structure, corresponding to an imaginary or complex wavenumber. This exponentially decreasing field is analogous to the evanescent wave in total reflection.

Thus, from the linear dispersion relation we find a necessary condition for the velocity of the localized steady structures: it must be complementary to the linear phase velocities. (This complementarity was also emphasized by Flierl 1987.) The second step is to determine this velocity in some independent way. In many cases this can be done from an integral relation giving the velocity of the centre of mass. Much attention has been given to this problem for planetary flows (cf. Cushman–Roisin, Chassignet & Tang 1990; Nycander & Sutyrin 1992 and references therein), but a systematic derivation for the conditions of the experiments appears not to have been done before.

Previous experience from a large number of cases indicates that if a suitable integral relation for the centre-of-mass velocity can be found, and if it shows that the velocity can be outside the region of linear phase velocities, then steady localized solutions exist.

This applies, for instance, to the Korteweg–de Vries (KdV) equation, to the modified KdV equation, and to many two-dimensional nonlinear equations describing various plasma modes of hydrodynamic type, such as nonlinear drift waves (Nycander 1991) and magnetic electron modes (Nycander & Pavlenko 1991).

Most importantly, it applies to solitary vortex solutions of the shallow-water equations on a β-plane, as shown by Nycander & Sutyrin (1992). In that paper explicit steady vortices were found by perturbation analysis, using a circular vortex with arbitrary radial profile as the zeroth-order solution. The β-effect and the propagation velocity were included to first order, resulting in a small deformation of the circular shape. Because of the basic scaling used, the vortex had to be larger than the Rossby radius.

The properties of this solution could be easily understood from the analysis described above. The amplitude indeed decreases exponentially outward if the vortex propagates westward faster than the fastest linear Rossby waves. Furthermore, the integral condition that determines the centre-of-mass velocity corresponds exactly to a solvability condition in the explicit solution, connecting the propagation velocity of the vortex and the zeroth-order radial profile.

These results were confirmed by numerical simulations. Simulations were also performed where the first-order solution was dropped, choosing instead a purely circular monopole vortex as the initial condition. After an adjustment period the vortex velocity described gentle oscillations around the value for the truly steady solution. Apparently, the steady vortices found by perturbation analysis are both stable and attracting.

The equations to be studied in the present paper are very similar to the shallow-water equations used by Nycander & Sutyrin (1992). The main difference is that because of the centrifugal force in a rotating paraboloidal vessel, the effective gravity g^* (cf. figure 1) depends on the latitude (i.e. on the distance from the bottom of the vessel). It is clear from previous work that the influence of this γ-effect is best seen from the way in which it affects the dispersion relation for linear Rossby waves, and the integral relation for the centre-of-mass velocity. Knowing that, it is easy to predict how it will affect the existence of steady vortices, without actually calculating such solutions explicitly. Thus, we will confine ourselves to deriving the linear dispersion relation and the relevant integral relation, with complete confidence that explicit steady solutions can also be found, similarly to the calculations by Nycander & Sutyrin.

3. Derivation of the shallow-water equations in paraboloidal coordinates

The derivation in this section is similar to the derivation of the shallow-water equations on a sphere by Pedlosky (1987). We start from the Euler equations, describing three-dimensional incompressible flow in a rotating coordinate system:

$$\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -2\omega_0\,\hat{\boldsymbol{z}}\times\boldsymbol{v}-\frac{\nabla p}{\rho_0}, \tag{1}$$

$$\nabla\cdot\boldsymbol{v} = 0. \tag{2}$$

Here $\omega_0\hat{\boldsymbol{z}}$ is the angular velocity of the system and p the pressure perturbation. For simplicity we neglect viscosity. (The effect of Ekman friction will be briefly considered at the end of §5.) The equilibrium pressure p_0 is

$$p_0 = \rho_0(\tfrac{1}{2}\omega_0^2 r^2 - gz)+C, \tag{3}$$

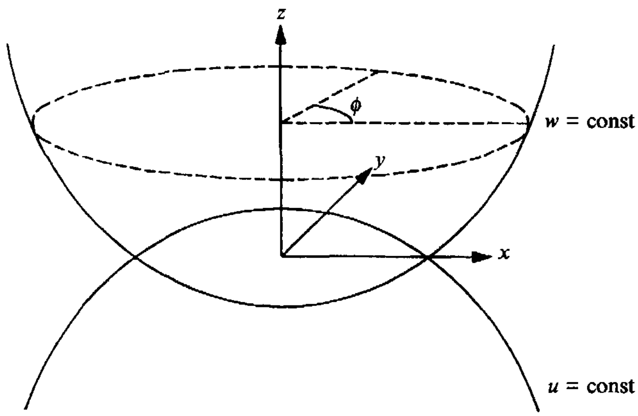

FIGURE 2. Paraboloidal coordinates.

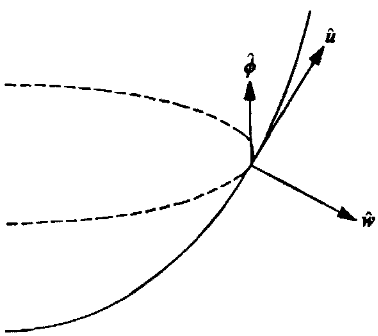

FIGURE 3. Orthogonal unit vectors in paraboloidal coordinates.

where r is the distance from the rotation axis, g the acceleration due to gravity and C a constant.

We will transform to the paraboloidal coordinates (u, w, ϕ) defined by (Spiegel 1968)

$$x = uw\cos\phi, \quad y = uw\sin\phi, \quad z = \tfrac{1}{2}(u^2 - w^2).$$

(See figure 2). The gradient operator in these coordinates is written

$$\nabla = \frac{\hat{\boldsymbol{u}}}{(u^2+w^2)^{\frac{1}{2}}}\frac{\partial}{\partial u} + \frac{\hat{\boldsymbol{w}}}{(u^2+w^2)^{\frac{1}{2}}}\frac{\partial}{\partial w} + \frac{\hat{\boldsymbol{\phi}}}{uw}\frac{\partial}{\partial \phi}.$$

The unit vectors $\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{\phi}}$ are shown in figure 3.

The coordinate system is chosen so that the unperturbed water surface is given by $w = w_0 = \text{const}$. In cylindrical coordinates the equation for this surface is

$$z = \tfrac{1}{2}(r^2/w_0^2 - w_0^2).$$

Thus, w_0^2 is equal to the radius of curvature $R_c = g/\omega_0^2$ at the bottom of the paraboloid. Assuming that the pressure is zero at the surface, the equilibrium pressure can then be written

$$p_0 = \tfrac{1}{2}\rho_0 g(1 + u^2/w_0^2)(w^2 - w_0^2). \tag{4}$$

We now assume that the shallow-water parameter,

$$\delta = H/L, \tag{5}$$

is small, and that the Rossby number,

$$\epsilon = V/L2\omega_0, \tag{6}$$

is not large (but not necessarily smaller either). Here H is the depth of the water (measured perpendicularly to the surface, cf. figure 1), L is the typical lengthscale of the motion, and V is the typical fluid velocity. To lowest order in δ one finds from the w-component of (1) that the pressure perturbation is independent of the depth, $\partial p/\partial w = 0$ (cf. Pedlosky 1987). The pressure perturbation is determined by the condition that the total pressure must be zero at the perturbed surface. A simple calculation shows that this leads to the equation

$$p = \rho_0 g^*(u)\,\Delta H, \tag{7}$$

where we have introduced the effective gravity,

$$g^* = g(1+u^2/w_0^2)^{\frac{1}{2}} = g(1+r^2/R_c^2)^{\frac{1}{2}}. \tag{8}$$

The perturbation of the fluid depth ΔH is measured along the w-axis, perpendicularly to the unperturbed surface. Equation (7) is the hydrostatic approximation: the pressure is given by the weight (in the effective gravity field) of the fluid.

We are now ready to write the equations of motion in paraboloidal coordinates. Inserting (7) into (1), and neglecting the w-component of the velocity, we obtain

$$\frac{\mathrm{d}v_u}{\mathrm{d}t} - \frac{v_\phi^2}{u(u^2+w_0^2)^{\frac{1}{2}}} = f(u)\,v_\phi - \frac{1}{(u^2+w_0^2)^{\frac{1}{2}}}\frac{\partial}{\partial u}[g^*(u)\,\Delta H], \tag{9}$$

$$\frac{\mathrm{d}v_\phi}{\mathrm{d}t} + \frac{v_\phi v_u}{u(u^2+w_0^2)^{\frac{1}{2}}} = -f(u)\,v_u - \frac{1}{uw_0}\frac{\partial}{\partial \phi}[g^*(u)\,\Delta H], \tag{10}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_\perp\cdot\boldsymbol{\nabla}_\perp = \frac{\partial}{\partial t} + \frac{v_u}{(u^2+w_0^2)^{\frac{1}{2}}}\frac{\partial}{\partial u} + \frac{v_\phi}{uw_0}\frac{\partial}{\partial\phi}, \tag{11}$$

and the Coriolis parameter is defined by

$$f = \frac{2\omega_0}{(1+u^2/w_0^2)^{\frac{1}{2}}}. \tag{12}$$

We have set $w \approx w_0$, since $\Delta H \ll R_c$ according to the shallow-water approximation.

Finally, we need the equation of continuity in the shallow-water approximation. It is derived similarly as for spherical symmetry (Pedlosky 1987), using the kinematic condition at the bottom (i.e. at the surface of the vessel) and at the free fluid surface. The result is

$$\frac{\mathrm{d}H}{\mathrm{d}t} + H\boldsymbol{\nabla}_\perp\cdot\boldsymbol{v}_\perp = 0, \tag{13}$$

where $\mathrm{d}/\mathrm{d}t$ is defined in (11), and

$$\boldsymbol{\nabla}_\perp\cdot\boldsymbol{v}_\perp = \frac{1}{(u^2+w_0^2)^{\frac{1}{2}}}\frac{\partial v_u}{\partial u} + \frac{v_u}{u(u^2+w_0^2)^{\frac{1}{2}}} + \frac{1}{uw_0}\frac{\partial v_\phi}{\partial\phi}.$$

Equations (9), (10) and (13) are the shallow-water equations in paraboloidal coordinates. With dissipation included, they should be the most realistic two-dimensional equations capable of describing the experiments with rotating vessels. Note that the only parameter that was assumed to be small in the derivation was the shallow-water parameter δ. The Rossby number ϵ is allowed to be of order unity (but not large), as is the ratio α between the typical lengthscale of the flow and the radius of curvature (or the size) of the vessel:

$$\alpha = L/R_c. \tag{14}$$

Below, we will assume that both ϵ and α are also small. It is then convenient to introduce new coordinates with the dimension length:

$$x = \phi u_0 w_0, \tag{15}$$

$$y = -(u-u_0)(u_0^2+w_0^2)^{\frac{1}{2}}, \tag{16}$$

where u_0 is a 'typical' value of u (i.e. approximately the u-coordinate of the vortex centre). Note that y increases downward in the vessel, toward the bottom point, which is the analogue of the north pole of a planet. We also introduce the notation $v_\phi = v_x$ and $-v_u = v_y$. Substituting this into (9), (10) and (13), and expanding to first order in $\alpha \sim y/R_c$, we obtain

$$\frac{dv_x}{dt} - \kappa_1 v_x v_y = f(y) v_y - (1+\kappa_1 y)\frac{\partial}{\partial x}[g^*(y)\Delta H], \tag{17}$$

$$\frac{dv_y}{dt} + \kappa_1 v_x^2 = -f(y) v_x - (1+\kappa_2 y)\frac{\partial}{\partial y}[g^*(y)\Delta H], \tag{18}$$

$$\frac{\partial H}{\partial t} + (1+\kappa_1 y)\frac{\partial}{\partial x}(v_x H) + (1+\kappa_2 y)\frac{\partial}{\partial y}(v_y H) - \kappa_1 v_y H = 0, \tag{19}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (1+\kappa_1 y) v_x \frac{\partial}{\partial x} + (1+\kappa_2 y) v_y \frac{\partial}{\partial y}, \tag{20}$$

$$f = \frac{2\omega_0}{(1+u_0^2/w_0^2)^{\frac{1}{2}}}(1+\kappa_2 y), \tag{21}$$

$$g^* = g(1+u_0^2/w_0^2)^{\frac{1}{2}}(1-\kappa_2 y), \quad \kappa_1 = \frac{1}{u_0(u_0^2+w_0^2)^{\frac{1}{2}}}, \quad \kappa_2 = \frac{u_0}{(u_0^2+w_0^2)^{\frac{3}{2}}}. \tag{22}$$

Equations (17)–(19) are the basic equations to be used in the following.

4. The linear dispersion relation

As the first step in finding out whether (17)–(19) can support steady monopole vortex solutions, we calculate the linear dispersion relation. The equations describe both high-frequency gravity waves and low-frequency Rossby waves. Since the vortex dynamics is characterized by a small Rossby number ϵ, we are mainly interested in the low-frequency branch, and therefore solve the linearized equations by expanding in ϵ. This means that the gravity waves are excluded from the analysis. The possibility that the vortex loses energy by coupling to gravity waves will be briefly considered at the end of this section.

Strictly speaking, (17)–(19) cannot be Fourier transformed in the y-direction, since the coefficients are y-dependent. However, if $\alpha \sim \kappa_{1,2} y \ll 1$ we may calculate the local dispersion relation in the geometric-optics approximation. We also allow for a weak y-dependence of the equilibrium depth $H_0(y)$, of the same magnitude as that of $f(y)$ and $g^*(y)$. Thus, linearizing and expanding to lowest order in ϵ and α, we obtain

$$\omega = \frac{\eta-\beta}{1+\rho_R^2 k^2} f \rho_R^2 k_x, \tag{23}$$

where the Rossby radius is defined by $\rho_R^2 = H_0 g^*/f^2$, and

$$\eta = H_0'(y)/H_0, \tag{24}$$

$$\beta = f'(y)/f. \tag{25}$$

The essential point here is that the γ-effect (the y-dependence of the effective gravity) does not affect the linear dispersion relation to lowest order, which is therefore identical to the dispersion relation for planetary Rossby waves.

Equation (23) shows that the phase velocity ω/k_x of the linear Rossby waves is confined to the interval $-v_{\max} < \omega/k_x < 0$, where

$$v_{max}(y) = (\beta-\eta) g^* H_0/f. \tag{26}$$

We note that the maximum phase velocity v_{max} is y-dependent. Thus, even if the vortex velocity is larger than the local value of v_{max}, there is a region some distance away where the Rossby waves can travel faster than the vortex. The condition for stationarity is that this region should be sufficiently far away that the coupling to the linear waves is insignificant. In §6 this condition will be made more precise, and used for estimating the amplitude necessary for stationarity.

We finally briefly consider the coupling to gravity waves, which are governed by the high-frequency branch of the dispersion relation:

$$\omega^2 = f^2 + g^* H_0 k^2, \tag{27}$$

where the inhomogeneities included in (17)–(19) have been neglected. Thus, the minimum phase velocity v_0 of the gravity waves is $v_0 = (g^* H_0)^{\frac{1}{2}}$. This should be compared with the propagation velocity of the vortex, which is approximately equal to the maximum phase velocity v_{max} of the Rossby waves, as will be shown in §5. Assuming that $H_0 = \text{const.}$, i.e. $\eta = 0$, and evaluating (26) using (8) and (12), we obtain

$$v_{max} = \tfrac{1}{2} H_0 (g/R_c)^{\frac{1}{2}} \frac{r}{(R_c^2 + r^2)^{\frac{1}{2}}}, \tag{28}$$

We find that $v_0/v_{max} > 2(R_c/H_0)^{\frac{1}{2}} \gg 1$. Thus, the gravity waves as described by the shallow-water equations always travel much faster than the vortex, and there can be no coupling.

If we go beyond the shallow-water approximation, the phase velocity can be lower, as seen from the dispersion relation $\omega^2 = gk$ for surface waves on deep water. However, in order to satisfy the resonance condition the wavelength of the gravity waves must be much smaller than the depth of the water, i.e. very much smaller than the radius of the vortex. The coupling is therefore very weak.

This is an important difference between geophysical flows and the experiments. On a planet we have $\omega_0 \ll (g/R)^{\frac{1}{2}}$ instead of the equality $\omega_0 = (g/R_c)^{\frac{1}{2}}$. (Here ω_0 is the angular velocity of the planet, and R is its radius.) The resonance condition with gravity waves must therefore be evaluated from case to case. In the atmosphere of the

Earth, for instance, we find that the phase velocity $(gH_0)^{\frac{1}{2}}$ obtained from (27) is only slightly larger than the propagation velocity of the vortex, and the coupling to gravity waves is therefore a stronger effect than in the experiments.

5. Centre-of-mass velocity

The first step in calculating the centre-of-mass velocity is to find the integral relation expressing mass conservation. Because of the geometric corrections in (19) we have $(\mathrm{d}/\mathrm{d}t)\int \Delta H\,\mathrm{d}x\,\mathrm{d}y \neq 0$. Instead, mass conservation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int [1-(\kappa_1+\kappa_2)\,y]\,\Delta H\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{29}$$

We have here neglected terms that are quadratic in $\kappa_1 y$ and $\kappa_2 y$, since they are smaller by two orders in the size parameter α than the leading terms. (To obtain a correct result to that order one would have to include many more terms in (17)–(19).) The reason for the correction term in (29) is that x and y are not uniform Cartesian coordinates, so that the real, physical size of the surface element $\mathrm{d}x\,\mathrm{d}y$ is smaller at larger values of y.

Using (19), we then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int x[1-(\kappa_1+\kappa_2)\,y]\,\Delta H\,\mathrm{d}x\,\mathrm{d}y = \int (1-\kappa_2 y)\,v_x\,H\,\mathrm{d}x\,\mathrm{d}y. \tag{30}$$

where we have again neglected terms that are α^2 times smaller than the leading term. We have also assumed that the fields ΔH, v_x and v_y decrease sufficiently fast at infinity, so that all boundary terms from the partial integrations vanish.

Equation (30) attains a more useful form if v_x can be expressed in terms of ΔH. We therefore solve (17) and (18) by expansion in ϵ, setting $\boldsymbol{v} = \boldsymbol{v}^{(0)}+\boldsymbol{v}^{(1)}$. To lowest order we neglect the left-hand side, and obtain

$$v_x^{(0)} = -\frac{1+\kappa_2 y}{f}\frac{\partial}{\partial y}(g^*\Delta H). \tag{31}$$

The next-order solution is

$$\begin{aligned} v_x^{(1)} &= -\frac{1}{f}\left[\frac{\partial v_y^{(0)}}{\partial t}+(1+\kappa_1 y)\,v_x^{(0)}\frac{\partial v_y^{(0)}}{\partial x}+(1+\kappa_2 y)\,v_y^{(0)}\frac{\partial v_y^{(0)}}{\partial y}+\kappa_1(v_x^{(0)})^2\right] \\ &= \frac{g^*}{f^2}(1+2\kappa_1 y)\frac{\partial^2}{\partial x^2}(v_x^{(0)}H)+\frac{g^*}{f^2}(1+\kappa_1 y+\kappa_2 y)\frac{\partial^2}{\partial x\,\partial y}(v_y^{(0)}H)-\frac{\kappa_1 g^*}{f^2}\frac{\partial}{\partial x}(v_y^{(0)}H) \\ &\quad -\frac{g^*}{f^2}(1+2\kappa_1 y)\,v_x^{(0)}\frac{\partial^2 H}{\partial x^2}-\frac{1+\kappa_2 y}{f}v_y^{(0)}\frac{\partial}{\partial y}\left[\frac{g^*}{f}(1+\kappa_1 y)\frac{\partial H}{\partial x}\right]-\frac{\kappa_1}{f}(v_x^{(0)})^2, \end{aligned} \tag{32}$$

where we have again neglected terms which are a factor α^2 smaller than the leading terms.

The lowest-order contribution to the centre-of-mass velocity is obtained by inserting (31) into (30):

$$\frac{\mathrm{d}^{(0)}}{\mathrm{d}t}\int x[1-(\kappa_1+\kappa_2)\,y]\,\Delta H\,\mathrm{d}x\,\mathrm{d}y = \int\frac{g^*}{f}[H_0'(y)\,\Delta H-\beta(H_0+\tfrac{1}{2}\Delta H)\,\Delta H-\gamma\tfrac{1}{2}(\Delta H)^2]\,\mathrm{d}x\,\mathrm{d}y, \tag{33}$$

where $$\gamma = g^{*\prime}(y)/g^*, \tag{34}$$

and β is defined in (25). Except for the γ-effect (the last term of (33)), this is the same result as that obtained for planetary flows (Nycander & Sutyrin 1992). From (21) and (22) we see that $\beta = \kappa_2$ and $\gamma = -\kappa_2$, so that the γ-effect exactly cancels the nonlinear part of the β-effect. Defining the centre of mass by

$$R = \int r[1-(\kappa_1+\kappa_2)y]\,\Delta H\,\mathrm{d}x\,\mathrm{d}y \Big/ \int [1-(\kappa_1+\kappa_2)y]\,\Delta H\,\mathrm{d}x\,\mathrm{d}y, \tag{35}$$

we obtain

$$\frac{\mathrm{d}R_x}{\mathrm{d}t} = \frac{g^* H_0}{f}(\eta-\beta), \tag{36}$$

where η is defined in (24). We have here neglected terms of the kind $\int y \Delta H\,\mathrm{d}x\,\mathrm{d}y$, assuming that the vortex is circular to lowest order. These terms are then a factor α^2 smaller than the leading-order terms, i.e. of the same magnitude as terms that have already been neglected.

According to (36) the centre-of-mass velocity is independent of the amplitude, and coincides with the phase velocity $-v_{max}$ of the fastest linear waves, unlike planetary flows. This result is the same as for the quasi-geostrophic vorticity equation (Nycander & Sutyrin 1992), and seems to imply that steady monopole vortices are impossible. Since even a small correction to this result would be important, it makes sense to carry out the calculation to the next order in ϵ. This correction is obtained by inserting (32) into (30). After some calculations we find

$$\frac{\mathrm{d}^{(1)}}{\mathrm{d}t}\int x[1-(\kappa_1+\kappa_2)y]\,\Delta H\,\mathrm{d}x\,\mathrm{d}y = \int \frac{g^{*2}H}{f^3}\left\{-\beta\left(\frac{\partial \Delta H}{\partial x}\right)^2 + \kappa_1\left[\left(\frac{\partial \Delta H}{\partial x}\right)^2 - \left(\frac{\partial \Delta H}{\partial y}\right)^2\right]\right\}\mathrm{d}x\,\mathrm{d}y. \tag{37}$$

The second term on the right-hand side of (37) (i.e. the term proportional to κ_1) is proportional to the ellipticity of the vortex. For a vortex that is circular to lowest order, with the non-circularity being proportional to the 'inhomogeneity parameters' η, β, γ, κ_1 and κ_2, the magnitude of this term is a factor α^2 smaller than the first term in (37). It can therefore be neglected to the same degree of accuracy as used throughout. (Vortices in shear flows, on the other hand, are often strongly elliptic, and this term can then be important.)

Finally, adding the contributions from (33) and (37), (36) is replaced by

$$\frac{\mathrm{d}R_x}{\mathrm{d}t} = \frac{g^* H_0}{f}\left(\eta - \beta - \beta\frac{g^*}{H_0 f^2}\frac{\int H(\partial \Delta H/\partial x)^2\,\mathrm{d}x\,\mathrm{d}y}{\int \Delta H\,\mathrm{d}x\,\mathrm{d}y}\right). \tag{38}$$

Equation (38) contains a new nonlinear contribution to the centre-of-mass velocity. Comparing with (26) we see that anticyclones propagate faster than the linear Rossby waves, while the velocity of cyclones is within the region of linear phase velocities. Thus, anticyclones but not cyclones can be steady and localized. Qualitatively, this is the same conclusion as for planetary flows, but in the laboratory experiments with rotating vessels the nonlinear contribution is much smaller because of the γ-effect.

The simple form of the nonlinear correction in (38) indicates that it should be possible to find a simple physical interpretation of it. This is most easily done from the shallow-water equations on the β-plane. These are a simplified version of (17)–(19), where we neglect all geometric corrections, but let the Coriolis parameter, the effective gravity and the equilibrium depth be variable. In other words, we set $\kappa_1 = \kappa_2 = 0$, while

f, g^* and H_0 are functions of y. (This is not a completely consistent approximation, but it may still be useful.) For the resulting set of equations the following exact integral relation holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int [Hv_y + xf(y)\,\Delta H]\,\mathrm{d}x\,\mathrm{d}y = \int [Hv_y\, xf'(y) + g^*(y)\,H_0'(y)\,\Delta H - g^{*\prime}(y)\tfrac{1}{2}(\Delta H)^2]\,\mathrm{d}x\,\mathrm{d}y.$$

The integrand on the left-hand side is the analogue of the generalized momentum of a charged particle in a magnetic field (Nycander 1990). For a steady and almost circular solution propagating with the velocity $U\hat{x}$ this can be reduced to

$$U\int_0^\infty f(y)\,\Delta H r\,\mathrm{d}r = \int_0^\infty [\tfrac{1}{2}Hv_\theta rf'(y) + g^*(y)\,H_0'(y)\,\Delta H - g^{*\prime}(y)\tfrac{1}{2}(\Delta H)^2]\,r\,\mathrm{d}r,$$

where v_θ is the azimuthal velocity, and we have used the fact that $\langle\cos^2\theta\rangle = \frac{1}{2}$ when integrating over θ. We then substitute v_θ from the equation of motion, expressing the Coriolis force through the pressure gradient and the centrifugal force. After some partial integrations, and neglecting terms that are quadratic in the equilibrium gradients, we obtain

$$U = \frac{g^*H_0}{f}(\eta-\beta) - \frac{(\beta+\gamma)\,E_p + \beta E_k}{fM}, \tag{39}$$

where we have introduced the potential energy, $E_p = \frac{1}{2}\int g^*(\Delta H)^2\,\mathrm{d}x\,\mathrm{d}y$, the kinetic energy, $E_k = \frac{1}{2}\int Hv_\theta^2\,\mathrm{d}x\,\mathrm{d}y$, and the mass anomaly $M = \int \Delta H\,\mathrm{d}x\,\mathrm{d}y$. The total energy $E_p + E_k$ is exactly conserved by the shallow-water equations on the β-plane. Notice that we have not assumed that the Rossby number is small in the derivation of (39).

The first term on the right-hand side of (39) is equal to the maximum velocity of the linear Rossby waves while the second term is the crucial nonlinear contribution. For planetary flows, when $\gamma = 0$, this nonlinear term is proportional to the total energy of the vortex divided by the mass anomaly. To lowest order in the Rossby number we can neglect the kinetic energy, and obtain the result in Nycander & Sutyrin (1992). In the experiments with rotating paraboloidal vessels, on the other hand, $\beta+\gamma = 0$, and the potential energy vanishes from (39). The nonlinear contribution to the velocity is therefore entirely due to the kinetic energy E_k. The last term of (38) is the geostrophic approximation of this contribution.

So far dissipative terms have been neglected, and therefore $\mathrm{d}R_y/\mathrm{d}t = 0$. However, the effect of bottom friction (i.e. the viscous dissipation in the Ekman layer) may easily be incorporated by adding the Ekman pumping $h_E\,\Omega$ to the right-hand side of (19) (Dolzhanskii, Krymov & Manin 1990). Here $\Omega = \partial_x v_y - \partial_y v_x$ is the vorticity, and $h_E = (\nu/2f)^{\frac{1}{2}}$ is the thickness of the Ekman layer where ν is the viscosity. The main effect of this term is spindown and dissipative widening of the vortex. The widening rate is most easily calculated for vortices on the f-plane (i.e. setting $\kappa_1 = \kappa_2 = 0$ and neglecting the y-dependence of all the equilibrium parameters f, g^* and H_0). Using (19), inserting the velocity $\boldsymbol{v}^{(0)}$ in the dissipative term and $\boldsymbol{v}^{(0)} + \boldsymbol{v}^{(1)}$ in the other terms, and using the f-plane approximation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int r^2\,\Delta H\,\mathrm{d}x\,\mathrm{d}y = 4h_E\frac{g^*}{f}\int \Delta H\,\mathrm{d}x\,\mathrm{d}y.$$

Defining the characteristic radius a of the vortex as $a = (\langle r^2\rangle)^{\frac{1}{2}}$, where

$$\langle r^2\rangle = \int r^2\,\Delta H\,\mathrm{d}x\,\mathrm{d}y \Big/ \int \Delta H\,\mathrm{d}x\,\mathrm{d}y,$$

this can be written

$$\frac{\mathrm{d}a}{\mathrm{d}t}=\frac{2h_E g^*}{a\ f}. \tag{40}$$

For the quasi-geostrophic vorticity equation this result is easily generalized to the β-plane. The only difference is that r above is then the distance from a point which moves with the centre-of-mass velocity $-\beta\hat{x}$. For the shallow-water equations on the β-plane with Ekman pumping included it is more difficult, since the velocity of the centre of mass depends on the amplitude, but (40) should still give a good estimate of the widening rate.

The Ekman friction also causes a drift in the y-direction, which may be calculated as $-\beta h_E g^*/f$ using the β-plane equations. However, comparing with (40) we see that this velocity is a factor $\beta a \sim \alpha$ smaller than the widening rate, and therefore too small to be of any significance.

6. Estimate of the amplitude necessary for stationarity

As pointed out in §4, the maximum phase velocity v_{max} of the Rossby waves is a function of latitude. It is therefore not enough for a steady vortex to propagate just slightly faster than the local value of v_{max}, since there would then exist a region nearby where Rossby waves with long wavelength travel faster than the vortex. If one tries to calculate an explicit steady solution with such a velocity, one finds that it is oscillatory in this region (Nycander & Sutyrin 1992), and that the energy of this oscillatory tail diverges (Nycander 1989). Dynamically, this means that a vortex with finite energy radiates energy by coupling with the linear waves. This phenomenon was seen in the simulation of the intermediate geostrophic dynamics by Matsuura & Yamagata (1982). (Linear waves were excited at the southern edge of the anticyclone, where Rossby waves travel faster than at its centre.) However, if the vortex propagates fast enough, the oscillatory region is far away, and the amplitude decreases exponentially in the intermediate region. The leakage of energy is then very small, and can be neglected for all practical purposes.

The criteria for the vortex to be considered steady are therefore that the distance L_{osc} to the oscillatory region should be larger than the radius of the vortex (the distance to the separatrix), and also much larger than the 'damping length' L_d (i.e. the inverse imaginary wavenumber) in the region between the vortex and the oscillatory region.

Assuming that H_0 is constant (i.e. $\eta = 0$), we can find the explicit dependence of the maximum phase velocity v_{max} on the distance r from the rotation axis of the vessel. From (28) we have $v_{max} = Cr/(r^2+R_c^2)^{\frac{3}{2}}$, where C is a constant. The relevant quantity, however, is the angular velocity v_a around the vessel, since the vortex travels on a fixed latitude (i.e. a constant value of r). Waves propagating higher up on the vessel travel a longer distance, and must therefore travel faster in order to keep up with the vortex. Correcting for this we obtain $v_a = C/(r^2+R_c^2)^{\frac{3}{2}}$, where C is another constant. Notice that v_a increases toward the bottom of the vessel, i.e. toward the 'north pole', so that the oscillating tail appears on the poleward side of the vortex. In the planetary case it appears on the equatorward side.

The angular propagation velocity of the vortex can be estimated from (38):

$$U_a \approx -v_a\left(1+\frac{\rho_R^2}{L^2}\frac{\bar{H}\,\overline{\Delta H}}{H_0^2}\right),$$

where $\overline{\Delta H}$ and $\bar{H}$ are characteristic values of the depth perturbation and the total depth

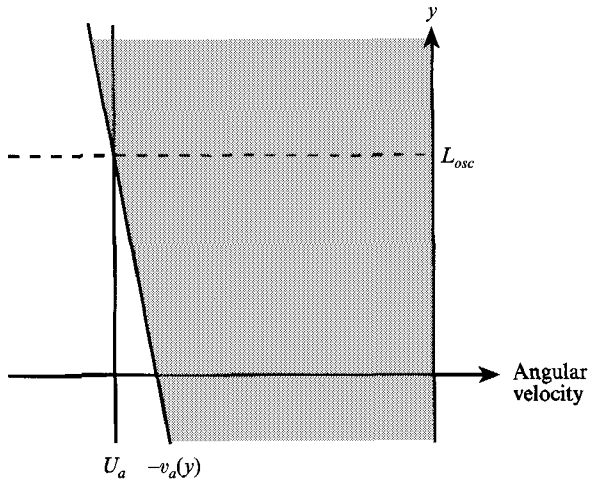

FIGURE 4. Illustration of the condition for stationarity. On the horizontal axis there is angular velocity around the vessel, while the latitude y is on the vertical axis, with the vortex centre at $y = 0$. The region of linear Rossby wave propagation is shaded, and U_a is the propagation velocity of the vortex. the difference between U_a and $-v_a$ at $y = 0$ is given by the last, nonlinear term on the right-hand side of (38). In the region $y > L_{osc}$ the resonance condition is satisfied, and linear waves are excited. For the vortex to be almost steady, this region must be far away.

in the vortex, and L is its characteristic radius. The distance to the region where U_a and the local value of $-v_a$ are equal (cf. figure 4) is then approximately

$$L_{osc} \approx \frac{\rho_R^2}{L^2} \frac{\bar{H}\overline{\Delta H}}{H_0^2} \frac{v_a}{\partial v_a/\partial y} = \frac{\rho_R^2}{L^2} \frac{\bar{H}\overline{\Delta H}}{H_0^2} \frac{(r^2+R_c^2)^{\frac{3}{2}}}{rR_c}.$$

For $r \approx R_c$ as in the experiments by Nezlin *et al.* (1990) (corresponding to mid-latitudes on a planet) we obtain

$$L_{osc} \approx \epsilon \frac{\bar{H}}{H_0} 3R_c, \tag{41}$$

where we introduced the Rossby number $\epsilon = \rho_R^2 \overline{\Delta H}/(L^2 H_0)$. The condition $L_{osc} > L$ can now be written

$$\epsilon \bar{H}/H_0 > \tfrac{1}{3}\alpha, \tag{42}$$

where the size parameter α is defined in (14). Using (8) and (12), (41) may be rewritten as

$$L_{osc} \approx 2 \frac{\bar{H}\overline{\Delta H} R_c^2}{H_0 L^2}, \tag{43}$$

Equation (42) is then replaced by

$$2\frac{\bar{H}}{L} > \frac{H_0}{\Delta H}\alpha^2. \tag{44}$$

The condition (42) gives a lower bound for the Rossby number, and (44) means that the shallow-water parameter δ, too, cannot be arbitrarily small.

To evaluate the other condition we must first estimate the 'damping length'. From the dispersion relation (23) we obtain the wavenumber in the small-amplitude region of any steady solution:

$$k^2 = -\frac{1}{\rho_R^2}\left(\frac{\beta f \rho_R^2}{U} + 1\right),$$

where U is the velocity in the x-direction of the solution, corresponding to the phase velocity ω/k_x of a linear wave. Estimating this velocity from (38), assuming that the last, nonlinear term is smaller than the linear one (i.e. we exclude vortices with $\Delta H \gg H_0$), we obtain the damping length $L_d = |k^2|^{-\frac{1}{2}}$:

$$L_d \approx \frac{LH_0}{(\bar{H}\overline{\Delta H})^{\frac{1}{2}}}.$$

The condition $L_{osc} \gg L_d$, which guarantees that the radiation of linear waves is insignificant, can then be written

$$\epsilon \bar{H}^{\frac{3}{2}} \overline{\Delta H}^{\frac{1}{2}} / H_0^2 \gg \tfrac{1}{3}\alpha. \tag{45}$$

This condition is more restrictive than (42), except when the amplitude is very large. It can also be rewritten as a lower bound for δ by using (43), similarly to the form (44).

These estimates may be compared to the experiments by Nezlin *et al.* (1990). They used a vessel rotating with the angular velocity $\omega_0 = 7.5$ s^{-1}, giving $R_c = 17$ cm, and the equilibrium depth H_0 was constant and between 1 and 5 cm. The parameters of the vortices in the extreme cases were approximately: (*a*) $H_0 = 1$ cm, $\Delta H/H_0 \approx 1$, $\rho_R \approx 4$ cm, $L \approx 2\rho_R$, and (*b*) $H_0 = 5$ cm, $\Delta H/H_0 \approx 0.2$, $\rho_R \approx 10$ cm, $L \approx \rho_R$.

In case (*a*) we get $L_{osc} \approx 25$ cm from (41), and $L_d \approx 0.7L \approx 6$ cm. Thus, the required inequalities are well satisfied, and these anticyclones should indeed be unaffected by Rossby-wave dispersion. The experimentally measured velocity, on the other hand, does not seem to be quite as large as predicted from (38).

In case (*b*) we get $L_{osc} \approx 10$ cm $\approx L$, and these vortices must radiate Rossby waves. However, cyclones should be much more strongly affected than anticyclones, since they are almost entirely within the oscillatory region. For the anticyclones this region starts somewhere near the separatrix. This can still explain the observed cyclone–anticyclone asymmetry.

Thus, the experimental observations appear compatible with the present theoretical results. However, the data are not precise enough to permit a detailed comparison, and cannot be said to confirm the theory. It should also be noted that the ratio $\alpha = L/R_c$ is approximately 0.5, which is not very small, and this of course limits the applicability of the theory. In particular, it means that the geometric optics approximation, which is the basis of the local relation (26), is not very accurate.

Vortex experiments in flat rotating vessels have been carried out by Kloosterziel & van Heijst (1991) and by Takematsu & Kita (1988). In these experiments the Rossby radius was larger than the whole vessel, i.e. much larger than in the experiments by Nezlin *et al.* (The Rossby radius is approximately equal to the geometric mean of the fluid depth H_0 and the radius of curvature of the fluid surface, R_c.) Vortices in such experiments either have zero circulation, in which case they are typically unstable, as demonstrated by Kloosterziel & van Heijst, or else they are poorly localized, with the azimuthal velocity inversely proportional to the distance from the vortex centre. In either case their propagation velocity is not determined by the integral relations (38) and (39), but turns out to be much lower. A simple way to understand this is to note that such flows can be described to a first approximation by the two-dimensional Euler equation (i.e. the barotropic vorticity equation with a rigid lid), and the maximum phase velocity of Rossby waves on the β-plane in the framework of this equation is infinite. Thus, the present theory does not apply to this parameter regime, which is very different from that studied by Nezlin *et al.*

7. Summary and discussion

In the previous sections the shallow-water equations appropriate to describe laboratory experiments in rotating paraboloidal vessels were derived in paraboloidal coordinates. The main difference to the corresponding equations for planetary flows is that the effective gravity g^* is variable in the laboratory experiments because of the centrifugal force (here called the γ-effect). These equations were then analysed in two ways.

First, the dispersion relations (23) and (27) for linear waves were derived. It was shown that the Rossby waves are not affected by the γ-effect, and that the gravity waves as described by the shallow-water equations cannot couple to the vortex.

Secondly, the integral relation (38) for the centre-of-mass velocity was derived. To lowest order in the Rossby number ϵ, it turns out that the γ-effect exactly cancels the nonlinear part of the β-effect. Thus, unlike for planetary flows, the centre-of-mass velocity coincides with the phase velocity of the fastest (long-wavelength) Rossby waves to this order. To rephase this result: the 'scalar nonlinearity' in the simplified equation used by Nezlin *et al.* (1990) vanishes identically.

To higher order in ϵ, however, another nonlinear contribution was found that causes anti-cyclones to propagate faster than cyclones, and faster than the Rossby waves. It was seen that this contribution is proportional to the kinetic energy of the vortex. It is of course also present for planetary flows, but there the dominant nonlinear contribution comes from the potential energy. (This contribution vanishes in the laboratory experiments owing to the γ-effect.)

Equation (38) was derived under the assumption that three parameters are small: the shallow-water parameter δ, the Rossby number ϵ, and the ratio α between the typical lengthscale of the flow and the radius of curvature of the vessel, defined in (5), (6) and (14), respectively. Unlike in many other theoretical treatments, however, no assumption was made about the ratio $\Delta H/H_0$ between the perturbation of the fluid depth and the equilibrium depth. In particular, the expression is valid even for $\Delta H \gg H_0$. (Such 'exotic' vortices were observed by Nezlin 1986, but apparently no systematic study of their properties has been made.) If we let $H_0 \rightarrow 0$ while keeping ΔH fixed, the two first (linear) terms on the right-hand side of (38) vanish, while the last term stays constant. Thus, in this limit it should be particularly simple to measure the effect of this important nonlinear term experimentally.

A similar case in $\eta = \beta$. As pointed out by Nezlin (1986), this is achieved when the equilibrium fluid surface and the surface of the vessel are described by exactly similar paraboloids, so that the equilibrium depth along the vertical z-axis (i.e. the rotation axis of the vessel) is constant. The depth H_0 (which is measured perpendicularly to the equilibrium surface, cf. figure 1) then decreases away from the rotation axis, similarly to the Coriolis parameter. Again, the propagation velocity of the vortex is then entirely given by the last term on the right-hand side of (38), and cyclones and anticyclones propagate in opposite directions.

In both these cases, $H_0 \rightarrow 0$ and $\eta = \beta$, the Rossby waves are suppressed, so that the existence condition for localized steady vortices discussed in §2 (i.e. that they must propagate faster than the Rossby waves) is trivially satisfied. To evaluate this condition in a less trivial situation, we have also considered the case when H_0 is constant, i.e. $\eta = 0$, in §6. The conditions (42) or (44) and (45) then determine the necessary amplitude for the vortex to be steady. If they are satisfied, the oscillatory region (where linear Rossby waves can propagate faster than the vortex) is so far away that the energy lost by radiation of Rossby waves is insignificant.

No steady solutions have been calculated explicitly here. However, there is no doubt that this can be done if the existence conditions mentioned above are satisfied, in a manner similar to the perturbation analysis by Nycander & Sutyrin (1992). A circular vortex with arbitrary radial profile is then chosen as the zeroth-order solution, neglecting the β-effect and the γ-effect. These inhomogeneous effects are taken into account in the first-order solution, and determine the propagation velocity of the vortex and cause in a slight deformation of its circular shape. This has also recently been confirmed by a numerical simulation of the general geostrophic equation on the β-plane (Nezlin & Sutyrin 1993). It showed that anticyclones can indeed propagate steadily even when the variable effective gravity is included.

From (38) it can be seen that the crucial nonlinear contribution to the centre-of-mass velocity is inversely proportional to L^2, i.e. to the area of the vortex. For vortices in planetary flows, on the other hand, the largest nonlinear contribution is independent of the size. Consequently, the existence criteria presented in §6 favour small vortices much more than the corresponding criteria for planetary flows (Nycander & Sutyrin 1992). We can therefore expect that steady vortices are in general smaller (as measured by the Rossby radius) in the experiments than in planetary flows.

Finally, one might speculate about further possible extension of the theory. The important relation (38) is based on an expansion to second order in ϵ, but only to first order in δ and α. From the symmetry of the calculations in §5 it may be seen that the next-order terms in α vanish for an almost circular vortex, like the term $\int y \Delta H \,\mathrm{d}x\,\mathrm{d}y$ already mentioned. Thus, the largest new terms that could be obtained from an expansion to higher order in α would be of order α^3. Such an expansion would not be meaningful, since the vortex can hardly be considered as a localized structure if these terms are not very small.

An expansion to higher order in the shallow-water parameter δ, on the other hand, might reveal some important new contribution. Such an expansion is also motivated by the condition (44), which shows that δ cannot be arbitrarily small in a vortex that propagates fast enough to be steady. In this case, however, the hydrostatic approximation (7) is not longer valid, which would complicate the calculations a great deal. No attempt has therefore been made to expand to higher order in δ.

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